Higher Dimensional Integrable Models with Painlevé Property Obtained from (1+1)-Dimensional Schwarz KdV Equation

Li-li Chen and Sen-yue Lou[a]

Department of Physics, Normal College, Ningbo University, Ningbo 315211, P. R. China
[a] Institute of Modern Physics, Normal College, Ningbo University, Ningbo 315211, P. R. China;
Institute of Theoretical Physics, Academia Sinica, P. O. Box 2735, Beijing 100080, P. R. China

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Using the extended Painlevé analysis, we obtained some higher dimensional integrable models with the Painlevé property from the (1+1)-dimensional Schwarz KdV equation.

The lower-dimensional soliton phenomena have widely been studied both by theory [1] and experiment [2, 3]. However, there are many difficulties in the study of the higher dimensional soliton theory. Recently, one of the present authors pointed out that, if one wants to find some higher dimensional integrable models, equations with conformal invariance (invariance under the Möbius transformation) [4] may be the most suitable candidates [5]. Some types of general higher dimensional Schwarz equations are proved to be integrable in the sence that they can be changed to some forms with the Painlevé property (PP) [6]. Now, one of the important problems is what kind of Schwarz equations can be obtained from the known physical equations. In [7] it is also shown that higher dimensional integrable models may also be obtained from lower dimensional ones. In [8], one of the present authors has extended Conte's invariant Painlevé analysis [9] to a *noninvariant* (under the Möbious transformation) but more generalized form in the *same dimensions* to get some exact solutions of various nonlinear equations by means of a nonstandard truncation approach. In this paper we extend Conte's Painlevé analysis to a different but still *invariant* form in *higher dimensions* such that we can find many higher dimensional integrable models from a known lower dimensional integrable model, say, the Schwarz KdV (SKdV) equation

$$\{\phi;\,x\}+\frac{\phi_t}{\phi_x}+\lambda=0,\ \left(\{\phi;\,x\}\equiv\frac{\phi_{xxx}}{\phi_x}-\frac{3}{2}\phi_{xx}^2\phi_x^2\right) \quad (1)$$

which is a variant form of the usual KdV equation.

Reprint requests to Dr. Sen-yue Lou;
E-mail: sylou@public.nbptt.zj.cn.

In order to show the PP of (1), we take the transformations

$$\phi=\exp\rho,\ \rho_x=u,\ \rho_t=g \quad (2)$$

at first. The final equations for u and g read

$$uu_{xx}-\frac{1}{2}u^4-\frac{3}{2}u_x^2+gu+\lambda u^2=0,\ u_t-g_x=0. \quad (3)$$

Equation (3) comes from the consistence condition of the transformation (2). The PP of (3) is obvious because of the KdV equation possesses the PP. However, in order to get more information from the Painlevé analysis, we assume that u and g are not only the functions of the explicit space-time variables $\{x,t\}$ but also the functions of the inner space variables $\{y,z\}$. It is known that, if a partial differential equation (PDE) possesses the PP, then the singular manifold ϕ in the usual Painlevé expansion is arbitrary. Because of the arbitrariness of the singular manifold, one can take some different types of forms, say, Conte changes the expansion function (singular manifold ϕ) as

$$\chi\equiv\left(\frac{\phi_x}{\phi}-\frac{\phi_{xx}}{2\phi_x}\right)^{-1}$$

such that all the coefficients in the new Painlevé expansion possess the Möbius transformation invariance [10]. In order to include the inner parameters in our further results, we can take

$$\xi\equiv\left(\frac{\phi_y}{\phi}-\frac{\phi_{yy}}{2\phi_y}\right)^{-1} \quad (4)$$

as a new expansion function. Differentiating (4) with respect to x and t, respectively, we can obtain two identities:

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$$\xi_x = p_1 - p_{1y}\xi + \frac{1}{2}(p_1 s + p_{1yy})\xi^2, \tag{5}$$

$$\xi_t = p_0 - p_{0y}\xi + \frac{1}{2}(p_0 s + p_{0yy})\xi^2, \tag{6}$$

where

$$p_0 \equiv \frac{\phi_t}{\phi_y}, p_1 \equiv \frac{\phi_x}{\phi_y}, s \equiv \{\phi;\ y\} \equiv \frac{\phi_{yyy}}{\phi_y} - \frac{3}{2}\frac{\phi_{yy}^2}{\phi_y^2} \tag{7}$$

are all conformal invariant, invariant under the Möbius transformation:

$$\phi \longrightarrow \frac{a\phi + b}{c\phi + d}.$$

With help of the leading order analysis of (3), we know that the functions $\{u,\ g\}$ should be expanded as

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-1}, \quad g = \sum_{j=0}^{\infty} g_j \phi^{j-1} \tag{8}$$

with

$$g_0^2 = p_0^2, \quad u_0 = g_0 p_1/p_0. \tag{9}$$

Substituting (8) into (3) and using (5) and (6), we have

$$(j+1)(j-1)u_0 u_j = f(u_i,\ g_i,\ i \le j-1,\ p_0,\ p_1, s), \tag{10}$$

$$(j-1)(g_j u_0 - u_j g_0) = u_{j-1,t} - g_{j-1,x}, \tag{11}$$

where f is a complicated function of $\{u_i,\ g_i,\ i \le j-1,\ p_0,\ p_1,\ s\}$ and its derivatives. From (10) and (11), it is not difficult to see that the resonances are located at

$$j = -1,\ 1,\ 1. \tag{12}$$

The resonance at $j = -1$ corresponds to the arbitrariness of the singularity manifold ξ (and then ϕ). At two resonances, $j = 1$, there are two compatibility conditions

$$-u_0^2 p_{1x} + u_0 u_{0x} p_1 - 2u_0^3 u_1 + 2u_1 u_0 p_1^2 = 0, \tag{13}$$

$$p_{1t} - p_0 p_{1y} - p_{0x} + p_1 p_{0y} = 0 \tag{14}$$

which must be satisfied. After substituting (9) and (7) into (13) and (14) one can see that (13) and (14) are satisfied identically. In the standard approach, in order to obtain the Bäcklund transformation of (3), one takes

$$\begin{aligned} &u_j(u_i, g_i, i \le j-1, p_0, p_1, s) = 0, \\ &g_j(u_i, g_i, i \le j-1, p_0, p_1, s) = 0,\ j \ge 2. \end{aligned} \tag{15}$$

Because (15) is conformal invariant for all $j \ge 2$ and the original Schwarz KdV equation (1) is integrable, we believe that every one of (15) is integrable. Even if this idea is not true for all j, it is still possible to get some higher dimensional integrable models from (15) by checking their PP for small j. Here we check the PP only for $u_2 = 0$ which has the concrete form

$$\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \lambda + 3\frac{\phi_x\phi_{xy}\phi_{yy}}{\phi_y^3} - 3\frac{\phi_x^2\phi_{yy}^2}{4\phi_y^4} - 3\frac{\phi_x\phi_{xyy}}{2\phi_y^2} + 3F_x - 3F^2 + 3F\left(\frac{\phi_x\phi_{yy}}{\phi_y^2} - \frac{\phi_{xx}}{\phi_x}\right) = 0, \tag{16}$$

where $F(\equiv u_1)$ is an arbitrary function. From (16), we see that the (2+1)-dimensional model will be reduced back to the original (1+1)-dimensional SKdV equation (1) when we drop the y-dependent terms or take $y \sim x$. Using the transformations

$$\phi = \exp\rho, \quad \rho_x = u, \quad \rho_y = v, \quad \rho_t = g, \tag{17}$$

(16) becomes

$$\frac{1}{4}u^4v^4 - \frac{3}{4}u^4v_y^2 - \frac{3}{2}v^4u_x^2 + uv^4u_{xx} - \frac{3}{2}u^3v^2u_{yy} + 3F_xu^2v^4 + \lambda u^2v^4 - 3F^2u^2v^4 + guv^4 + 3u^3vu_yv_y + 3u^3Fv^2v_y - 3Fuv^4u_x = 0, \tag{18}$$

$$u_t = g_x, \tag{19}$$

$$v_t = g_y. \tag{20}$$

It is easy to prove the PP of (18) - (20) by taking the traditional WTC (Weiss-Tabor-Carnevale) approach [10]. However, in order to get (3+1)-dimensional models with conformal invariance and the PP, we introduce

$$\eta \equiv \left(\frac{\phi_z}{\phi} - \frac{\phi_{zz}}{2\phi_z}\right)^{-1} \tag{21}$$

as a new expansion function. Differentiating (21) with respect to x, y and t, respectively, we can obtain the three identities

$$\eta_x = P_1 - P_{1y}\eta + \frac{1}{2}(P_1 S + P_{1yy})\eta^2, \tag{22}$$

$$\eta_y = P_2 - P_{2y}\eta + \frac{1}{2}(P_2 S + P_{2yy})\eta^2, \tag{23}$$

$$\eta_t = P_0 - P_{0y}\eta + \frac{1}{2}(P_0 S + P_{0yy})\eta^2, \tag{24}$$

where

$$P_0 \equiv \frac{\phi_t}{\phi_z},\ P_1 \equiv \frac{\phi_x}{\phi_z},\ P_2 \equiv \frac{\phi_y}{\phi_z},\ S \equiv \{\phi;\ z\}. \tag{25}$$

By means of the leading order analysis of (18) - (20), we know that the functions $\{u, v, g\}$ should be expanded as

$$u = \sum_{j=0}^{\infty} u_j \phi^{j-1}, v = \sum_{j=0}^{\infty} v_j \phi^{j-1}, g = \sum_{j=0}^{\infty} g_j \phi^{j-1} \tag{26}$$

with

$$g_0^2 = P_0^2,\ u_0 = P_1 \frac{g_0}{P_0},\ v_0 = P_2 \frac{g_0}{P_0}. \tag{27}$$

Substituting (26) into (18) - (20) and using (22) - (24), we have

$$(j+1)(j-1)(u_0 u_j + v_0 v_j) \tag{28}$$
$$= f(u_i, g_i, i \le j-1, P_0, P_1, P_2, S),$$

$$(j-1)(g_j u_0 - u_j g_0) = u_{j-1,t} - g_{j-1,x}, \tag{29}$$

$$(j-1)(g_j v_0 - v_j g_0) = v_{j-1,t} - g_{j-1,y}. \tag{30}$$

From (28) - (30) it is not difficult to see that the resonances are located at

$$j = -1,\ 1,\ 1,\ 1. \tag{31}$$

The three compatibility conditions at the resonances, $j = 1$, now read

$$-3u_0^3 u_1 v_0^2 P_2^2 + \frac{3}{2}u_0^4 v_0^2 P_{2y} + u_0^4 v_0^3 v_1 + 3F u_0^2 v_0^4 P_1 + 2v_0^3 v_1 u_0^2 P_1^2 - 3u_0^4 v_0 v_1 P_2^2 - 3F u_0^3 v_0^3 P_2 - \frac{3}{2}u_0^4 v_{0y} v_0 P_2 + u_0^3 u_1 v_0^4 = 0, \tag{32}$$

$$u_0 P_{0z} - g_{0x} + u_{0t} - g_0 P_{1z} = 0, \tag{33}$$

$$v_0 P_{0z} - g_{0y} + v_{0t} - g_0 P_{2z} = 0, \tag{34}$$

and these equations are satisfied identically because of (25) and (27). So the (2+1)-dimensional system (18) - (20) (and then its Schwarz form (16)) is integrable under the meaning that it possesses the PP. In the same way, if we take

$$u_j = u_j(P_0, P_1, P_2, S, u_1, v_1, g_1) = 0,$$
$$v_j = v_j(P_0, P_1, P_2, S, u_1, v_1, g_1) = 0, \tag{35}$$
$$g_j = g_j(P_0, P_1, P_2, S, u_1, v_1, g_1) = 0,\ (j \ge 2)$$

we can get the Bäcklund transformation for the system (18) - (20). Furthermore, it is interesting that every one of the equations (35) for a fixed j may be integrable because all these equations are conformal invariant and are derived from the integrable equation (16). The simplest one of them is given by $u_2 = 0$, i. e.,

$$\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \lambda + \frac{9}{2}\frac{\phi_x\phi_{zz}\phi_{xz}}{\phi_z^3} - \frac{3}{4}\frac{\phi_x^2\phi_{yy}^2}{\phi_y^4} + \frac{3}{8}\frac{\phi_x^2\phi_{zz}^2}{\phi_z^4} + 3\frac{\phi_x^2\phi_{yzz}}{\phi_z^2\phi_y} - \frac{3}{2}\frac{\phi_x\phi_{xyy}}{\phi_y^2} + 3\frac{\phi_x\phi_{xy}\phi_{yy}}{\phi_y^3} - \frac{9}{4}\frac{\phi_x\phi_{xzz}}{\phi_z^2} - 6\frac{\phi_x^2\phi_{yz}\phi_{zz}}{\phi_z^3\phi_y} + G = 0, \tag{36}$$

where the arbitrary function G is related to other arbitrary functions F, u_1, v_1 and g_1 by

$$G = \Big(-6u_1 + 6\frac{\phi_x}{\phi_y}v_1\Big)F + 3u_{1x} - 3u_1^2 + \frac{3}{2}\frac{\phi_x^2}{\phi_y^2}v_1^2 - \frac{3}{2}\frac{\phi_x}{\phi_t}u_{1t} + 3\frac{\phi_x}{\phi_y}u_1 v_1 + \frac{3}{2}\frac{\phi_x^2}{\phi_t\phi_y}v_{1t} - \frac{3}{2}\frac{\phi_x^2}{\phi_t\phi_y}g_{1y} - 3\frac{\phi_x}{\phi_y}u_{1y} - \frac{3}{2}\phi_x^2\phi_y^2 v_{1y} - 3\frac{\phi_{xx}}{\phi_x}u_1 - 3\frac{\phi_x^2\phi_{zz}}{\phi_z^2\phi_y}v_1 + \frac{3}{2}\frac{\phi_x}{\phi_t}g_{1x} + 3\frac{\phi_x\phi_{xy}}{\phi_y^2}v_1 + \frac{3}{2}\frac{\phi_x\phi_{zz}}{\phi_z^2}u_1 + \frac{3}{2}\frac{\phi_x\phi_{yy}}{\phi_y^2}u_1 + 3F_x - 3F^2 + 3\Big(\frac{\phi_x\phi_{yy}}{\phi_y^2} - \frac{\phi_{xx}}{\phi_x}\Big)F = 0. \tag{37}$$

If we drop y and z related terms or if fixed y and z are proportional to x, we reobtain the (1+1)-dimensional SKdV equation (1) from the (3+1)-dimensional equation (36).

We have proved the PP of (36) by using the Painlevé analysis and that some (4+1)-dimensional integrable models with the PP can be obtained when we introduce a further inner parameter into the procedure of proving the PP of the (3+1)-dimensional model (36). One of the simplest models obtained in this way reads

$$\begin{aligned}
&\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \lambda - \frac{3}{4}\frac{\phi_x^2\phi_{yy}^2}{\phi_y^4} + \frac{9}{2}\frac{\phi_x\phi_{zz}\phi_{xz}}{\phi_z^3} - \frac{9}{4}\frac{\phi_x\phi_{xzz}}{\phi_z^2} + 3\frac{\phi_x\phi_{xy}\phi_{yy}}{\phi_y^3} + \frac{3}{8}\frac{\phi_x^2\phi_{zz}^2}{\phi_z^4} - \frac{9}{2}\frac{\phi_x^2\phi_{x_4x_4}\phi_{yx_4}}{\phi_{x_4}^3\phi_y} \\
&+ \frac{9}{4}\frac{\phi_x^2\phi_{yx_4x_4}}{\phi_{x_4}^2\phi_y} - 6\frac{\phi_x^2\phi_{yz}\phi_{zz}}{\phi_z^3\phi_y} - \frac{3}{16}\frac{\phi_x^2\phi_{x_4x_4}^2}{\phi_{x_4}^4} + 3\frac{\phi_x\phi_{xx_4}\phi_{x_4x_4}}{\phi_{x_4}^3} + \frac{9}{4}\frac{\phi_x^2\phi_{x_4x_4}\phi_{zx_4}}{\phi_{x_4}^3\phi_z} - \frac{9}{8}\frac{\phi_x^2\phi_{zx_4x_4}}{\phi_{x_4}^2\phi_z} \\
&+ 3\frac{\phi_x^2\phi_{yzz}}{\phi_z^2\phi_y} - \frac{3}{2}\frac{\phi_x\phi_{xx_4x_4}}{\phi_{x_4}^2} - \frac{3}{2}\frac{\phi_x\phi_{xyy}}{\phi_y^2} = 0.
\end{aligned} \tag{38}$$

Equation (38) is a (4+1)-dimensional generalization of the SKdV (1). If we use some reductions on (38), say, $x_4 = z,\ x_4 = y,\ \ldots$, we can get some further (3+1)-dimensional SKdV equations.

Using the above procedure step by step, we can get various integrable models with the PP in any dimensions.

In summary, after introducing some inner parameters explicitly in the expansion variables, we can get many higher dimensional integrable models from lower dimensional ones. Some concrete SKdV equations with the PP in (2+1)-, (3+1)- and (4+1)-dimensions are listed here for further investigations.

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